Des. Codes Cryptogr. (2008) 47:75–97
DOI 10.1007/s10623-007-9075-4

A 51-dimensional embedding of the Ree–Tits generalized octagon

Kris Coolsaet

Received: 25 October 2006 / Revised: 7 March 2007 / Accepted: 2 April 2007 /
Published online: 5 May 2007

Abstract We construct an embedding of the Ree–Tits generalized octagon defined over a field K in a 51-dimensional projective space over K arising from a 52-dimensional Lie algebra $\mathbf{J}$ of type F_4. This construction derives from a quadratic map (related to a 'standard' duality of F_4) from the 26-dimensional module (see K. Coolsaet, Adv Geometry, to appear) into $\mathbf{J}$. (This embedding is full if and only if K is a perfect field.) We provide explicit formulas for the coordinates of the points of the octagon in this embedding, in terms of their Van Maldeghem coordinates. We apply these results to compute the dimensions of subspaces generated by various special subsets of points of the octagon: the sets of points at a fixed distance from a given point or a given line and the Suzuki suboctagons. The results depend on whether K is the field of 2 elements, or not.

Keywords Ree–Tits generalized octagon · Embedding

AMS Classification 51E12

1 Introduction

In 1960 J. Tits constructed the first examples of (thick) generalized octagons, using the discovery by R. Ree of the related twisted Chevalley groups of type ${}^2\mathsf{F}_4$ (in the perfect case) [6–8]. These *Ree–Tits* octagons are characterized as the only octagons satisfying the Moufang condition. No other thick generalized octagons are known, except for some 'free' and 'universal' constructions (which in some sense do not really count).

While there exists a lot of literature on embedding theory of generalized quadrangles and hexagons, for the octagons hardly anything is known. We hope that the description of the 51-dimensional embedding given here (together with that of the 25-dimensional embedding

K. Coolsaet (✉)
Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9,
Gent 9000, Belgium
e-mail: Kris.Coolsaet@UGent.be

in [1,2]) will make these rather unaccessible geometries easier to understand and will prove useful in future investigations.

That the Ree–Tits octagons with base field K can be embedded into a 25-dimensional projective space with the same base field, is a consequence of the fact that all points and lines of a Ree–Tits octagon can be regarded as points and lines of a metasymplectic space $\mathcal{F}$ which is closely related to the Lie algebra of type F_4 over K (see for instance [5, Section 2.5]). The projective embedding derives from the 26-dimensional module for this algebra.

The same algebra also has a 52-dimensional module and this representation can be used to provide an embedding of the hyperlines and planes of $\mathcal{F}$ as points and lines of a 51-dimensional projective geometry. Because 'hyperline' and 'plane' are dual notions of, respectively, 'point' and 'line', and because in a sense the points and lines of the Ree–Tits octagons are absolute elements of this duality, it will be no surprise that this 51-dimensional space can also be used to represent the points of the Ree–Tits octagon.

As we did in [2] for the 25-dimensional embedding, we will establish explicit formulas for the coordinates of points of the Ree–Tits octagon in this 51-dimensional space, in terms of their Van Maldeghem coordinates. We will also derive the corresponding subdimensions and study the subspaces generated by Suzuki suboctagons.

In Sect. 2 we will establish notations and review some of the material from [2] which is needed for this text. Section 3 defines the 51-dimensional embedding and explains how to compute the tables of Appendix A of projective coordinates for this embedding. Section 4 shows how these tables can be used to provide an embedding also in the non-perfect case. Finally, we derive the dimensions of subspaces of points at a fixed distance of a given point or line (Sect. 5) and of the subspace generated by the Suzuki suboctagons (Sect. 6).

2 Preliminaries

Recall that the elements of a root system Φ of type F_4 can be expressed as 4-tuples of real coordinates in the following way [4] :

1. There are 24 roots whose coordinates are permutations of 4-tuples of the form $(\pm 1, \pm 1, 0, 0)$.
2. There are 16 roots with coordinates of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$.
3. There are eight roots whose coordinates are permutations of 4-tuples of the form $(\pm 1, 0, 0, 0)$.

Roots with Euclidian length $\sqrt{2}$ are called *long* roots, roots with length 1 are called *short* roots. We shall borrow the shorthand notation from [1], writing $\bar{1}$ for -1, $+$ for $\frac{1}{2}$ and $-$ for $-\frac{1}{2}$. (Examples of roots are $0\bar{1}\bar{1}0$, $+--+$ or 0001.)

The root system Φ can be used to establish a 52-dimensional Lie-algebra $\mathbf{J}$ over a field K. $\mathbf{J}$ can be written as a direct sum

$$\mathbf{J} = \mathbf{G} \oplus \bigoplus_{r\in\Phi} K E_r,$$

with 48 basis elements E_r, one for each root r of Φ, and a 4-dimensional subspace $\mathbf{G}$ which is the so-called *Cartan subalgebra* or *torus* of $\mathbf{J}$. The Cartan subalgebra is generated by a set of elements H_r with $r \in \Phi$, a basis of which is provided by choosing four values for r that correspond to a fundamental system of Φ. We will use the following basis :

$$H_1 \stackrel{\text{def}}{=} H_{1\bar{1}00}, \quad H_2 \stackrel{\text{def}}{=} H_{01\bar{1}0}, \quad H_3 \stackrel{\text{def}}{=} H_{0010}, \quad H_4 \stackrel{\text{def}}{=} H_{---+}.$$

We will use the notation $A[r]$, $r \in \Phi$, and $A[1], \dots, A[4]$ for the coordinates of $A \in \mathbf{J}$ in terms of this basis. Hence $A = \sum A[r]E_r + A[1]H_1 + \cdots + A[4]H_4$.

Henceforth we will assume that the characteristic of K is 2. In this special case the elements H_3, H_4 and E_s where s is restricted to the short roots, generate a 26-dimensional subalgebra of $\mathbf{J}$. This subalgebra is a 26-dimensional $\mathbf{J}$-module. It turns out to be more convenient to work with an isomorphic copy $\mathbf{W}$ of this $\mathbf{J}$-module which is disjoint from $\mathbf{J}$. The basis elements of $\mathbf{W}$ will be denoted by h_3, h_4 and e_s. (The isomorphism maps upper case symbols to the equivalent lower case symbols.) We will use the notation $a[s]$ and $a[3]$, $a[4]$, for the coordinates of $a \in \mathbf{W}$ in terms of this basis.

To construct the geometry $\mathcal{F}$ of type F_4 we have introduced in [1,3] the notion of *isotropic* elements of $\mathbf{W}$ and of $\mathbf{J}$ (the latter are called *totally* isotropic in [1]). *Points* of $\mathcal{F}$ are 1-spaces Ke that correspond to isotropic elements e of $\mathbf{W}$. Likewise, *hyperlines* of $\mathcal{F}$ are 1-spaces KE that correspond to isotropic elements E of $\mathbf{J}$. A point Ke is incident with a hyperline KE if and only if $e \in \mathbf{W}E$, i.e., when e is an image of the action of the Lie algebra element E on some element of the module $\mathbf{W}$.

In characteristic 2 we may define a duality operation $Q(\cdot)$ which maps every isotropic element e of $\mathbf{W}$ onto an isotropic element $Q(e)$ of $\mathbf{J}$. This duality extends to a duality of the geometry $\mathcal{F}$ in the following sense: if Ke, Kf are points of $\mathcal{F}$ such that Ke is incident with $KQ(f)$, then $KQ(e)$ is incident with the point Kf^{frob}, where $\cdot^{\mathrm{frob}}$ denotes the action of the *Frobenius map* on $\mathbf{W}$ which squares every coordinate of f.

To establish an explicit formula for $Q(e)$ we make use of the fact that the set of short roots and the set of long roots are root systems in their own right, both of type D_4. There is a linear map $r \mapsto r^\dagger$ which provides an isomorphism between them. This map preserves the angles between roots, but not their length. It satisfies $r^{\dagger\dagger} = 2r$. The following table lists the images of all short roots, for easy reference.

r	$r^\dagger$	r	$r^\dagger$	r	$r^\dagger$
1000	$\bar{1}001$	++++	0101	----	$0\bar{1}0\bar{1}$
$\bar{1}000$	$100\bar{1}$	+++-	$\bar{1}100$	---+	$1\bar{1}00$
0100	0110	++-+	0011	--+-	$00\bar{1}\bar{1}$
$0\bar{1}00$	$0\bar{1}\bar{1}0$	++--	$\bar{1}010$	--++	$10\bar{1}0$
0010	$01\bar{1}0$	+-++	$00\bar{1}1$	-+-+	1010
$00\bar{1}0$	$0\bar{1}10$	+-+-	$\bar{1}0\bar{1}0$	-+--	$001\bar{1}$
0001	1001	+--+	$0\bar{1}01$	-++-	$010\bar{1}$
$000\bar{1}$	$\bar{1}00\bar{1}$	+---	$\bar{1}\bar{1}00$	-+++	1100

(1)

Now, let e be an isotropic element of $\mathbf{W}$. Then the coordinates of $Q = Q(e) \in \mathbf{J}$ are given by

$$\begin{aligned}
&Q[t^\dagger] = e[t]^2, \ \text{when } t^\dagger \in \Phi_L, && Q[t] = \sum_{\substack{\{u,v\}\subset\Phi_S \\ u+v=t^\dagger}} e[u]e[v], \ \text{when } t \in \Phi_S, \\
&Q[1] \ = e[4]^2, && Q[3] = \sum_{\substack{u\in\Phi_S \\ \langle 1\bar{1}00,u\rangle=-1}} e[u]e[-u], \\
&Q[2] \ = e[3]^2, && Q[4] = \sum_{\substack{u\in\Phi_S \\ \langle 1001,u\rangle=-1}} e[u]e[-u].
\end{aligned} \tag{2}$$

(The value for $Q[4]$ corrects the one given in [1], cf. Appendix B.)

To ensure the existence of a Ree–Tits octagon, we shall henceforth assume that the field K has a *Tits* endomorphism. i.e., an endomorphism σ satisfying $(k^\sigma)^\sigma = k^2$ for all $k \in K$. The corresponding Ree–Tits octagon is denoted by $\mathsf{O}(K,\sigma)$.

To construct the embedding $\mathcal{O}(K)$ of $\mathsf{O}(K,\sigma)$ into a 25-dimensional projective space, we proceed as follows: points of the Ree–Tits octagon are those points Ke of $\mathcal{F}$ such that Ke is incident with the hyperline $KQ(e)^{\sigma/2}$. (Note that $\sigma/2 = \sigma^{-1}$.) This definition only makes sense when $Q(e)^{\sigma/2}$ is known to exist, for instance when K is a perfect field (i.e., when $K^2 = K^\sigma = K$). We shall postpone the discussion of the case where K is not perfect to Sect. 4.

3 The 51-dimensional embedding in terms of Van Maldeghem coordinates

We are now in a position to define the embedding $\mathcal{O}'(K)$ of $\mathsf{O}(K,\sigma)$ which we shall use throughout this paper (we leave out the reference to the field K if it is clear from context).

Theorem 1 *Let K denote a perfect field of characteristic 2 with Tits automorphism σ. Let $\mathcal{O}$ denote the 25-dimensional embedding of $\mathsf{O}(K,\sigma)$ as described above. Define $\mathcal{O}'$ to be the set of 1-spaces of the form $KQ(e)^{\sigma/2}$ where Ke is a point of $\mathcal{O}$. Then $\mathcal{O}'$ provides a full embedding of $\mathsf{O}(K,\sigma)$ into a projective space of dimension 51, mapping points of $\mathsf{O}(K,\sigma)$ onto hyperlines of the metasymplectic space $\mathcal{F}$.*

Proof Note that $Q(e)^{\sigma/2}$ is isotropic when e is isotropic, and hence $KQ(e)^{\sigma/2}$ is a hyperline of $\mathcal{F}$. $Q(e)^{\sigma/2}$ belongs to the 52-dimensional vector space $\mathbf{J}$ and hence $KQ(e)^{\sigma/2}$ belongs to the 51-dimensional projective space associated with it.

It only remains to be proved that lines of $\mathsf{O}(K,\sigma)$ are mapped onto full lines of this projective space. This is an immediate consequence of [2, Proposition 6.2] or [3, Proposition 4.12], which state that $Q(ke + lf) = Q(ke) + Q(lf) = k^2Q(e) + l^2Q(f)$ whenever Ke, Kf are collinear points of $\mathcal{O}$. It follows that $Q(ke + lf)^{\sigma/2} = k^\sigma Q(e)^{\sigma/2} + l^\sigma Q(f)^{\sigma/2}$ and hence the image of every point on the line joining Ke and Kf lies on the line joining their images. Because the field is perfect, we have $K^\sigma = K$ and therefore also every point on the line joining the images $KQ(e)^{\sigma/2}$ and $KQ(f)^{\sigma/2}$ is the image of some point on the original line. □

The points of $\mathsf{O}(K,\sigma)$ can also be described by means of a coordinatization by Van Maldeghem [5]. This coordinatization assigns the coordinate (∞) to a fixed reference point of $\mathsf{O}(K,\sigma)$ and a unique coordinate of the form $(a, l, a', l', \ldots)$, with 1, 3, 5 or 7 entries, or $(k, b, k', b', \ldots)$, with 2, 4, or 6 entries, to every other point. The parameters $a, a', \ldots, b, b' \ldots$ range over the field K. The parameters $k, k', \ldots, l, l', \ldots$ denote pairs $k = (k_0, k_1), k' = (k'_0, k'_1), \ldots$of field elements. It is customary (cf. [5, Sections 3.6.1–3.6.2]) to treat these pairs as elements of a group $K_\sigma^{(2)}$ with composition $\oplus$ (non-Abelian in general), endowed with a trace and a norm operation with the following definitions:

$$k \oplus l \stackrel{\text{def}}{=} (k_0 + l_0, k_1 + l_1 + k_0^\sigma l_0)$$
$$T(k) \stackrel{\text{def}}{=} k_0^{1+\sigma} + k_1,$$
$$N(k) \stackrel{\text{def}}{=} k_0^{2+\sigma} + k_0k_1 + k_1^\sigma = k_0T(k) + k_1^\sigma = k_0k_1 + T(k)^\sigma.$$

We shall call the number of coordinates of a particular point its *arity* (the reference point has arity 0 by definition). The arity of a point is related to the distance of that point to the reference point and to a reference line (with Van Maldeghem coordinates $[\infty]$) which contains the reference point and all points with coordinates of arity 1. Points with odd arity i lie at distance i from the reference line and at distance $i+1$ from the reference point. Points with even arity i lie at distance i from the reference point and at distance $i+1$ from the reference line.

In [2] Van Maldeghem coordinates are used to establish an explicit link between points of $\mathsf{O}(K,\sigma)$ and the corresponding points in the embedding $\mathcal{O}$. A point in $\mathcal{O}$ with Van Maldeghem coordinates (∞) $((a,l,\ldots)$, $(k,b,\ldots)$, respectively) is denoted by Kp, $(Kp(a,l,\ldots)$, $Kp(k,b\ldots)$, respectively) and explicit formulas are given for the coordinates in $\mathbf{W}$ of the elements $p(\cdots)$ in terms of the parameters $a,b,k,l,\ldots$. In this paper we will do the same for the embedding $\mathcal{O}'$ of Theorem 1.

For ease of notation we denote elements of $K\times K_\sigma^{(2)}\times K\times\cdots$ and $K_\sigma^{(2)}\times K\times K_\sigma^{(2)}\times\cdots$ by symbols of the form $\vec{x}$. Van Maldeghem coordinates of a point are abbreviated to $(\vec{x})$ and the corresponding element of $\mathbf{W}$ is represented as $p(\vec{x})$. We also allow $\vec{x}$ to denote the symbol ∞ in which case $p(\vec{x})$ is understood to represent p.

Write $P(\vec{x})\stackrel{\text{def}}{=}Q(p(\vec{x}))^{\sigma/2}$. It is our intention to express each $P(\vec{x})$ as a linear combination of basis elements of $\mathbf{J}$. The results are listed in Appendix A. They were obtained by computer, but can be verified by hand using techniques similar to those of [2].

There are (at least) two different ways to obtain these results. The most straightforward method consists of applying (2) to the results of [2] and then apply $\sigma/2$. From the left column of (2) we see that 26 of the coordinates of $P(\vec{x})$ are essentially the same as those of the $p(\vec{x})$, More exactly, if $t\in\Phi_S$, then the coordinate of $P(\vec{x})$ at position $t^\dagger$ can be found by applying σ to the coordinate at position t of $p(\vec{x})$ (with the same parameters). Coordinates at positions 1 and 2 of $P(\vec{x})$ can be obtained in the same way from coordinates at positions 4 and 3 of $p(\vec{x})$. In other words, the 51-dimensional embedding $\mathcal{O}'(K)$ can be projected onto the 25-dimensional embedding $\mathcal{O}(K^\sigma)$ (which for perfect fields K is the same as $\mathcal{O}(K)$).

A second method consists of mimicking the techniques of [2]. This method uses certain elements $y(a)$, $y(k)$, w and S of the automorphism group ${}^2\widehat{\mathsf{F}}_4(K)$ of $\mathsf{O}(K,\sigma)$ (with $a\in K, k\in K_\sigma^{(2)}$) to derive points of a given arity from those of a smaller arity. Applying [2, Lemma 1] and [2, Lemma 2] to our case, yields

$$\begin{aligned}
P(a,l,a',l',a'',l'',a''') &= P(0,l,a',l',a'',l'',a''')\cdot y(a)\\
P(k,b,k',b',k'',b'') &= P(0,b,k',b',k'',b'')\cdot y(k)\\
P(a,l,a',l',a'') &= P(0,l,a',l',a'')\cdot y(a)\\
P(k,b,k',b') &= P(0,b,k',b')\cdot y(k)\\
P(a,l,a') &= P(0,l,a')\cdot y(a)\\
P(k,b) &= P(0,b)\cdot y(k)\\
P(a) &= P(0)\cdot y(a)
\end{aligned}$$

and

$$\begin{aligned}
P(0,l,a',l',a'',l'',a''') &= P(l,a',l',a'',l'',a''')\cdot Sw\\
P(0,b,k',b',k'',b'') &= P(b,k',b',k'',b'')\cdot S\\
P(0,l,a',l',a'') &= P(l,a',l',a'')\cdot Sw\\
P(0,b,k',b') &= P(b,k',b')\cdot S\\
P(0,l,a') &= P(l,a')\cdot Sw\\
P(0,b) &= P(b)\cdot S\\
P(0) &= p\cdot Sw
\end{aligned}$$

Table 1 Images of basis elements of $\mathbf{J}$ through $y(a)$, $a \in K$

E_{++++}	$E_{----} + aE_{\bar{1}000} + a^{\sigma} E_{++--} + a^{1+\sigma} E_{0100}$
E_{+++-}	$E_{---+} + a^{\sigma} E_{++-+}$
E_{++-+}	$E_{--+-} + a^{\sigma} E_{+++-}$
$E_{++--} + aE_{0100}$	$E_{--++} + a^{\sigma} E_{++++}$
E_{-+--}	E_{+-++}
E_{-+-+}	$E_{+-+-} + aE_{0010}$
E_{-++-}	$E_{+--+} + aE_{0001}$
E_{-+++}	$E_{+---} + aH_4 + a^2 E_{-+++}$
$E_{1000} + aE_{++++}$	$E_{\bar{1}000} + a^{\sigma} E_{0100}$
E_{0100}	$E_{0\bar{1}00} + aE_{--++} + a^{\sigma} E_{1000} + a^{1+\sigma} E_{++++}$
E_{0010}	$E_{00\bar{1}0} + aE_{-+-+}$
E_{0001}	$E_{000\bar{1}} + aE_{-++-}$
H_4	$H_3 \quad + aE_{-+++}$
E_{1100}	$E_{\bar{1}\bar{1}00} + a^{2\sigma} E_{1100} + a^{\sigma} H_1 + a^{\sigma} H_3$
$E_{\bar{1}100}$	$E_{1\bar{1}00} + aE_{+-++} + a^2 E_{0011}$
E_{1010}	$E_{\bar{1}0\bar{1}0} + a^{\sigma} E_{01\bar{1}0}$
$E_{\bar{1}010} + a^{\sigma} E_{0110}$	$E_{10\bar{1}0} + aE_{++-+} + a^2 E_{0101}$
E_{1001}	$E_{\bar{1}00\bar{1}} + a^{\sigma} E_{010\bar{1}}$
$E_{\bar{1}001} + a^{\sigma} E_{0101}$	$E_{100\bar{1}} + aE_{+++-} + a^2 E_{0110}$
E_{0110}	$E_{0\bar{1}\bar{1}0} + aE_{---+} + a^{\sigma} E_{10\bar{1}0} + a^{1+\sigma} E_{++-+} + a^2 E_{\bar{1}001} + a^{2+\sigma} E_{0101}$
$E_{01\bar{1}0}$	$E_{0\bar{1}10} + a^{\sigma} E_{1010}$
E_{0101}	$E_{0\bar{1}0\bar{1}} + aE_{--+-} + a^{\sigma} E_{100\bar{1}} + a^{1+\sigma} E_{+++-} + a^2 E_{\bar{1}010} + a^{2+\sigma} E_{0110}$
$E_{010\bar{1}}$	$E_{0\bar{1}01} + a^{\sigma} E_{1001}$
E_{0011}	$E_{00\bar{1}\bar{1}} + aE_{-+--} + a^2 E_{\bar{1}100}$
$E_{001\bar{1}}$	$E_{00\bar{1}1}$
$H_1 \quad + aE_{-+++}$	$H_2 \quad + a^{\sigma} E_{1100}$

for $a, a', a'', a''' \in K, l, l', l'' \in K_{\sigma}^{(2)}$ for $a, a', a'', a''' \in K, l, l', l'' \in K_{\sigma}^{(2)}$.

The group elements $y(a)$ and $y(k)$ are linear transformations that act on $\mathbf{J}$ as listed in Tables 1 and 2. In both tables the image of E_r is the entry that has E_r as its leading term.

The linear transformation w permutes coordinates according to the following scheme ($E_r \cdot w = E_{r'}$ when $r \to r'$ is in the table below)

$$0001 \to +--+ \to 0\bar{1}00 \to ---- \to 000\bar{1} \to -++- \to 0100 \to ++++ \to 0001$$
$$+-++ \to ---+ \to +--- \to --+- \to -+-- \to +++- \to -+++ \to ++-+ \to +-++$$
$$1000 \to --++ \to 00\bar{1}0 \to +-+- \to \bar{1}000 \to ++-- \to 0010 \to -+-+ \to 1000$$

$$1001 \to 0\bar{1}01 \to 0\bar{1}\bar{1}0 \to 0\bar{1}0\bar{1} \to \bar{1}00\bar{1} \to 010\bar{1} \to 0110 \to 0101 \to 1001$$
$$00\bar{1}1 \to 1\bar{1}00 \to \bar{1}\bar{1}00 \to 00\bar{1}\bar{1} \to 001\bar{1} \to \bar{1}100 \to 1100 \to 0011 \to 00\bar{1}1$$
$$\bar{1}001 \to 10\bar{1}0 \to 01\bar{1}0 \to \bar{1}0\bar{1}0 \to 100\bar{1} \to \bar{1}010 \to 0\bar{1}10 \to 1010 \to \bar{1}001$$

Table 2 Images of basis elements of **J** through $y(k)$, $k \in K_\sigma^{(2)}$

$E_{++++}+k_0E_{++-+}+k_1E_{+-++}+N(k)E_{+--+}$	E_{----}
$E_{+++-}+k_0E_{++--}+k_1E_{+-+-}+N(k)E_{+---}$	E_{---+}
$E_{++-+}+k_0^\sigma E_{+-++}+T(k)E_{+--+}$	$E_{--+-}+k_0E_{----}$
$E_{++--}+k_0^\sigma E_{+-+-}+T(k)E_{+---}$	$E_{--++}+k_0E_{---+}$
$E_{-+--}+k_0^\sigma E_{--+-}+T(k)E_{----}$	$E_{+-++}+k_0E_{+--+}$
$E_{-+-+}+k_0^\sigma E_{--++}+T(k)E_{---+}$	$E_{+-+-}+k_0E_{+---}$
$E_{-++-}+k_0E_{-+--}+k_1E_{--+-}+N(k)E_{----}$	E_{+--+}
$E_{-+++}+k_0E_{-+-+}+k_1E_{--++}+N(k)E_{---+}$	E_{+---}
E_{1000}	$E_{\bar{1}000}$
$E_{0100}+k_0^\sigma E_{0010}+T(k)H_3+k_1^\sigma E_{00\bar{1}0}$ $+N(k)^\sigma E_{0\bar{1}00}$	$E_{0\bar{1}00}$
$E_{0010}+k_0H_3+k_0^2E_{00\bar{1}0}+T(k)^\sigma E_{0\bar{1}00}$	$E_{00\bar{1}0}+k_0^\sigma E_{0\bar{1}00}$
E_{0001}	$E_{000\bar{1}}$
$H_4 \ +k_0E_{00\bar{1}0}+k_1E_{0\bar{1}00}$	H_3
$E_{1100}+k_0^\sigma E_{1010}+T(k)E_{1000}+k_1^\sigma E_{10\bar{1}0}$ $+N(k)^\sigma E_{1\bar{1}00}$	$E_{\bar{1}\bar{1}00}$
$E_{\bar{1}100}+k_0^\sigma E_{\bar{1}010}+T(k)E_{\bar{1}000}+k_1^\sigma E_{\bar{1}0\bar{1}0}$ $+N(k)^\sigma E_{\bar{1}\bar{1}00}$	$E_{1\bar{1}00}$
$E_{1010}+k_0E_{1000}+k_0^2E_{10\bar{1}0}+T(k)^\sigma E_{1\bar{1}00}$	$E_{\bar{1}0\bar{1}0}+k_0^\sigma E_{\bar{1}\bar{1}00}$
$E_{\bar{1}010}+k_0E_{\bar{1}000}+k_0^2E_{\bar{1}0\bar{1}0}+T(k)^\sigma E_{\bar{1}\bar{1}00}$	$E_{10\bar{1}0}+k_0^\sigma E_{1\bar{1}00}$
E_{1001}	$E_{\bar{1}00\bar{1}}$
$E_{\bar{1}001}$	$E_{100\bar{1}}$
$E_{0110}+k_0E_{0100}+k_1E_{0010}+k_0^2E_{01\bar{1}0}$ $+k_1^2E_{0\bar{1}10}+k_1N(k)E_{0\bar{1}00}+k_0N(k)E_{00\bar{1}0}$ $+N(k)^2E_{0\bar{1}\bar{1}0}+T(k)^\sigma H_2+N(k)H_3$	$E_{0\bar{1}\bar{1}0}$
$E_{01\bar{1}0}+k_0^\sigma H_2+k_0^{2\sigma}E_{0\bar{1}10}$ $+T(k)E_{00\bar{1}0}+k_0^\sigma T(k)E_{0\bar{1}00}+T(k)^2E_{01\bar{1}0}$	$E_{0\bar{1}10}+k_0E_{0\bar{1}00}+k_0^2E_{0\bar{1}\bar{1}0}$
$E_{0101}+k_0^\sigma E_{0011}+T(k)E_{0001}+k_1^\sigma E_{00\bar{1}1}$ $+N(k)^\sigma E_{0\bar{1}01}$	$E_{0\bar{1}0\bar{1}}$
$E_{010\bar{1}}+k_0^\sigma E_{001\bar{1}}+T(k)E_{000\bar{1}}+k_1^\sigma E_{00\bar{1}\bar{1}}$ $+N(k)^\sigma E_{0\bar{1}0\bar{1}}$	$E_{0\bar{1}01}$
$E_{0011}+k_0E_{0001}+k_0^2E_{00\bar{1}1}+T(k)^\sigma E_{0\bar{1}01}$	$E_{00\bar{1}\bar{1}}+k_0^\sigma E_{0\bar{1}0\bar{1}}$
$E_{001\bar{1}}+k_0E_{000\bar{1}}+k_0^2E_{00\bar{1}\bar{1}}+T(k)^\sigma E_{0\bar{1}0\bar{1}}$	$E_{00\bar{1}1}+k_0^\sigma E_{0\bar{1}01}$
$H_1+k_0^\sigma E_{0\bar{1}10}+T(k)E_{0\bar{1}00}+k_1^\sigma E_{0\bar{1}\bar{1}0}$	$H_2+k_0E_{00\bar{1}0}+k_1E_{0\bar{1}00}$

while S permutes E_r with $E_{r'}$ where r' is obtained from r by changing the sign of the middle two coordinates. (For example, $E_{++++}\cdot S = E_{+--+}$, $E_{1100}\cdot S = E_{1\bar{1}00}$, $E_{100\bar{1}}\cdot S = E_{100\bar{1}}$.)

The action of S and w on the torus $\mathbf{G}$ is as follows :

$$H_1 \cdot w = H_1 + H_3, \quad H_2 \cdot w = H_1 + H_2 + H_3,$$
$$H_3 \cdot w = H_3 + H_4, \quad H_4 \cdot w = H_4,$$

$$H_1 \cdot S = H_1, \quad H_2 \cdot S = H_2,$$
$$H_3 \cdot S = H_3, \quad H_4 \cdot S = H_4.$$

The explicit formulas for the actions of each of these group elements can be derived from the theory established in [1] or [3]. Alternatively they may be obtained from the corresponding actions on $\mathbf{W}$ which are listed in [2], using the fact that $Q(e^g) = Q(e)^g$ whenever $g \in {}^2\widehat{\mathsf{F}}_4(K)$ [1,3].

4 The case of a non-perfect field K

As in the case of the 25-dimensional embedding, the fact that the formulas in Appendix A make use of σ but never of σ^{-1} enables us to define the embedding $\mathcal{O}'$ also in the non-perfect case.

Theorem 2 *Let K be a field of characteristic 2, not necessarily perfect. Let σ be a Tits endomorphism of K. Then the map which maps a point with Van Maldeghem coordinates $(\vec{x})$ onto the point $KP(\vec{x})$, where $P(\cdots)$ is as given in the tables of Appendix A, is an embedding of $\mathsf{O}(K,\sigma)$ into the projective space associated with $\mathbf{J}$. This is a full embedding if and only if K is perfect.*

Proof (The proof is almost identical to that of [2, Theorem 5] so we shall only give a sketch here.)

The lines of $\mathsf{O}(K,\sigma)$ can be easily expressed in terms of Van Maldeghem coordinates. For example, the line with Van Maldeghem coordinates $[a,l,a',l',a'',l'']$, with $a,a',a'' \in K$, $l,l'l'' \in K_\sigma^{(2)}$ contains the points with coordinates (a,l,a',l',a'') and (a,l,a',l',a'',l'',a'''), for all $a''' \in K$. We may use the tables in Appendix A to verify that

$$P(a,l,a',l',a'',l'',a''') = P(a,l,a',l',a'',l'',0) + a'''^\sigma \cdot P(a,l,a',l',a'') \tag{3}$$

and hence this line is embedded as (part of) a line of the projective space associated with $\mathbf{J}$. Similar arguments hold for lines $[\cdots]$ of arity smaller than 6.

The only remaining lines are the lines with Van Maldeghem coordinates $[k,b,\ldots,k''']$ of arity 7. These contain the point with coordinates $(k,b,\ldots,b'')$ and the points with coordinates of the form $P(a,\Phi_6(a,k,b,\ldots,k'''),\ldots,\Phi_1(a,k,b,\ldots,k'''))$ with $a \in K$. (The functions Φ_i define the so-called octagonal octonary ring of the octagon. They can be expressed as explicit formulas in their arguments, involving σ but not σ^{-1}.)

In this case it is sufficient to prove that

$$\begin{aligned} &P(a,\Phi_6(a,k,b,\ldots,k'''),\ldots,\Phi_1(a,k,b,\ldots,k''')) \\ &= P(0,\Phi_6(0,k,b,\ldots,k'''),\ldots,\Phi_2(0,k,b,\ldots,k''')) + a^\sigma \cdot P(k,b,k',b',k'',b''), \end{aligned} \tag{4}$$

for all $a,b,\ldots,b'' \in K$, $k,\ldots,k''' \in K_\sigma^{(2)}$.

When K is perfect (4) can be obtained by applying $Q(\cdot)$ to the corresponding identity that holds in $\mathbf{W}$ (with P replaced by p and the factor a^σ by a). Therefore, when (4) is evaluated symbolically, the left hand side and the right hand side will yield identical expressions.

(This could also be verified by computer.) Because these expressions do not involve σ^{-1}, (4) remains valid when K is not a perfect field.

Finally, note that the scalar factors of the rightmost terms in both (3) and (4) range over K^σ and not K. It follows that lines of $\mathsf{O}(K, \sigma)$ will be embedded as full projective lines only if $K^\sigma = K$. □

5 Subdimensions

Define the (projective) *subdimension* d_i, for $i = 0, \ldots, 7$, to be the dimension of the projective space generated by the points of $\mathcal{O}'$ at distance i from a given point (for i even) or a line (for i odd). Because the Ree group acts transitively on the points and lines of $\mathcal{O}'$ and because the elements of the Ree group are induced by linear transformations of $\mathbf{J}$, this definition is independent of the chosen point or line. As before, we choose the point KP and the line connecting KP and $KP(0)$ (i.e., $KE_{0\bar{1}01} + KE_{1001}$) as reference point and reference line.

The points at distance i of the reference point or line are exactly those with arity i or $i-1$. Denote by $\mathbf{J}_i$ the subspace of $\mathbf{J}$ generated by the elements $P(\vec{x})$ having arity i or $i-1$. Then $d_i = \dim \mathbf{J}_i - 1$. It is easily seen that $\mathbf{J}_0 \le \mathbf{J}_1 \le \cdots \le \mathbf{J}_7 = \mathbf{J}_8$.

Lemma 3 *Let K be a field of characteristic 2 with Tits automorphism σ. If $|K| > 2$ then there exist $a, b, c \in K$ such that the determinant*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & a^\sigma & a^2 & a^{1+\sigma} \\ b & b^\sigma & b^2 & b^{1+\sigma} \\ c & c^\sigma & c^2 & c^{1+\sigma} \end{vmatrix} \tag{5}$$

is different from 0.

Proof Because $|K| > 2$ we may choose $a \neq 0, 1$. Consider $A = a + a^\sigma$. If $A = A^\sigma$, then $a + a^\sigma = a^\sigma + a^2$ and hence $a = a^2$, implying $a = 0$ or 1. Hence $A \neq A^\sigma$. In particular, $A \neq 0, 1$.

We shall prove that we can take $b = a^\sigma$ and $c = a + 1$. Then (5) reduces to

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & a^\sigma & a^2 & a^{1+\sigma} \\ a^\sigma & a^2 & a^{2\sigma} & a^{2+2\sigma} \\ a+1 & a^\sigma + 1 & a^2 + 1 & a^{1+\sigma} + a + a^\sigma + 1 \end{vmatrix} = (a + a^\sigma) \begin{vmatrix} 1 & 1 & 1 \\ a & a^\sigma & a^2 \\ a^\sigma & a^2 & a^{2\sigma} \end{vmatrix}.$$

(We have added the first two rows to the last one.) Now, adding the second column to the third and the first to the second, we find that this is equal to

$$A \begin{vmatrix} 1 & 0 & 0 \\ a & A & A^\sigma \\ a^\sigma & A^\sigma & A^2 \end{vmatrix} = A(A^3 + A^{2\sigma})$$

This is equal to zero if and only if $A = 0$ or $A^{2\sigma} = A^3$. Applying σ to the latter equality we obtain $A^4 = A^{3\sigma}$, and hence $A = A^4/A^3 = A^{3\sigma}/A^{2\sigma} = A^\sigma$, contradicting our assumption. □

Theorem 4 *The subdimensions $d_0, \ldots, d_7$ for the embedding $\mathcal{O}'$ of the Ree–Tits generalized octagon over a field K are as listed in the following table, where the values depend only on whether K is the finite field of order 2, or not.*

	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7
$K \neq \mathrm{GF}(2)$	0	1	5	13	29	43	50	51
$K = \mathrm{GF}(2)$	0	1	4	10	23	37	49	51

Proof By counting the number of canonical basis vectors of $\mathbf{J}$ that occur in the expressions for $P(\vec{x})$ for $\vec{x}$ of various arity (i.e., by counting the number of rows in each of the tables of Appendix 6) we easily arrive at upper bounds for the various subdimensions. These are exactly the numbers given in the first line of the table of subdimensions. Before we prove that these bounds can actually be attained when $|K| > 2$, we shall first discuss the special case $K = \mathrm{GF}(2)$.

In this case σ is the identity (and conversely, this is the only case in which the identity is a Tits endomorphism) and we find

$$a = a^\sigma, \quad T(k) = k_0 + k_1, \quad k_0 N(k) = k_0, \quad k_1 N(k) = k_1, \tag{6}$$

for all $a \in K, k \in K_\sigma^{(2)}$.

It is clear that $\mathbf{J}_0 = KE_{1001}$ and $\mathbf{J}_1 = KE_{1001} + KE_{0\bar{1}01}$. To establish a basis for $\mathbf{J}_2$, observe that the coefficients of $P(k, b)$ for $E_{00\bar{1}1}$, E_{0011} and E_{0001} are k_1^σ, k_0^σ and $T(k)$, which sum up to zero when $K = \mathrm{GF}(2)$, by (6), reducing the dimension of the corresponding subspace by 1. Hence the upper bound for d_2 can now be reduced to 4.

It is not so difficult to see that the following five elements may serve as a basis for $\mathbf{J}_2$ (still assuming that $K = \mathrm{GF}(2)$) :

$$E_{1001},\ E_{0\bar{1}01},\ E_{0101},\ E_{00\bar{1}1} + E_{0001},\ E_{0011} + E_{0001}.$$

Hence $d_2 = 4$.

(We will not give explicit derivations for the bases of the various $\mathbf{J}_i$. Although they are most easily computed by computer, it is still possible to obtain the results by hand.)

Likewise, the coefficients of $P(a, l, a')$ for $E_{10\bar{1}0}$, $E_{\bar{1}001}$, E_{---+} and E_{++-+} are $a^\sigma = a^2 = a = a^{1+\sigma}$, reducing the upper bound for d_3 by 3. We may compute the following basis for $\mathbf{J}_3$:

$$\begin{aligned}&E_{1001},\ E_{0\bar{1}01},\ E_{0101},\ E_{00\bar{1}1},\ E_{0011},\ E_{0001},\ E_{0\bar{1}\bar{1}0},\ E_{+--+},\\ &E_{1\bar{1}00},\ E_{+-++},\quad E_{10\bar{1}0} + E_{\bar{1}001} + E_{---+} + E_{++-+}.\end{aligned}$$

Similarly, for $P(k, b, k', b')$ we obtain

- The coefficients of $E_{01\bar{1}0}$, E_{0100} and $E_{00\bar{1}0}$ are $k_0^2 = k_0 = k_0 N(k)$.
- The coefficients of $E_{0\bar{1}10}$, E_{0010} and $E_{0\bar{1}00}$ are $k_1^2 = k_1 = k_1 N(k)$.
- The coefficient of H_2 is $T(k)^\sigma$, i.e., the sum of the two previous cases.
- The coefficients of E_{1010}, $E_{10\bar{1}0}$, $E_{\bar{1}001}$ and E_{1000}, i.e., $b^\sigma + k_0^\sigma k_0'^\sigma$, $b^\sigma k_0^2 + k_1^\sigma k_0'^\sigma$, b^2 and $b^\sigma k_0 + T(k) k_0'^\sigma$, sum up to zero. (Note that this is also true for the corresponding coefficients of $P(a, l, a')$.)

This reduces the upper bound for d_4 by 6. A corresponding basis of 24 elements is the following :

$$E_{1001},\ E_{0\bar{1}01},\ E_{0101},\ E_{00\bar{1}1},\ E_{0011},\ E_{0001},\ E_{0\bar{1}\bar{1}0},\ E_{+--+},$$
$$E_{1\bar{1}00}, E_{+-++},\ E_{0110},\ E_{++++},\ E_{1100},\ E_{++-+},\ E_{--++},\ E_{---+},$$
$$E_{-+-+},\ E_{-+++},\ H_3,$$
$$E_{10\bar{1}0}+E_{\bar{1}001},\ E_{10\bar{1}0}+E_{1000},\ E_{1010}+E_{1000},$$
$$E_{0\bar{1}10}+E_{0\bar{1}00}+E_{0010}+H_2,\ E_{01\bar{1}0}+E_{0100}+E_{00\bar{1}0}+H_2.$$

For $P(a,l,a',l',a'')$ we have

- The coefficients of $E_{100\bar{1}}, E_{\bar{1}010}, E_{--+-}$ and E_{+++-} are $a^\sigma = a^2 = a = a^{1+\sigma}$.
- The coefficients of $E_{\bar{1}0\bar{1}0}$ and E_{----} are $l_0^2 = l_0$.
- The coefficients of E_{++--} and $E_{\bar{1}000}$ are $a^\sigma l_0 = al_0$.
- The coefficients of $E_{01\bar{1}0}$ and E_{0100} are $a^\sigma l_0^2 = a^{1+\sigma} l_0$. (Note that this is also true for the corresponding coefficients of $P(k,b,k',b')$.)

This reduces the upper bound for d_5 by 6. A corresponding basis of 38 elements is given by

$$E_{1001},\ E_{0\bar{1}01},\ E_{0101},\ E_{00\bar{1}1},\ E_{0011},\ E_{0001},\ E_{0\bar{1}\bar{1}0},\ E_{+--+},$$
$$E_{1\bar{1}00}, E_{+-++},\ E_{0110},\ E_{++++},\ E_{1100},\ E_{++-+},\ E_{--++},\ E_{---+},$$
$$E_{-+-+},\ E_{-+++},\ H_3,\ E_{0\bar{1}0\bar{1}},\ E_{0\bar{1}00},\ E_{\bar{1}\bar{1}00},\ E_{1000},\ E_{10\bar{1}0},$$
$$E_{1010},\ E_{\bar{1}001},\ H_1,\ H_4,\ E_{0010},\ E_{+---},\ E_{0\bar{1}10},\ H_2,$$
$$E_{+-+-},\ E_{00\bar{1}0},\ E_{++--}+E_{\bar{1}000},\ E_{\bar{1}0\bar{1}0}+E_{----},\ E_{01\bar{1}0}+E_{0100},$$
$$E_{100\bar{1}}+E_{\bar{1}010}+E_{--+-}+E_{+++-}.$$

For $P(k,b,k',b',k'',b'')$ we find that the coefficients of $E_{001\bar{1}}, E_{00\bar{1}\bar{1}}$ and $E_{000\bar{1}}$, i.e., k_0^σ, k_1^σ and $T(k)$ again sum up to zero, decreasing the upper bound d_6 by 1. A basis of 50 elements is given by all canonical basis elements of $\mathbf{J}$, except $E_{\bar{1}00\bar{1}}, E_{001\bar{1}}, E_{00\bar{1}\bar{1}}$ and $E_{000\bar{1}}$, extended with the elements $E_{001\bar{1}}+E_{000\bar{1}}$, and $E_{00\bar{1}\bar{1}}+E_{000\bar{1}}$.

Finally, as both $E_{\bar{1}00\bar{1}}$ and $E_{000\bar{1}}$ are easily shown to belong to $\mathbf{J}_7$, we find that $\dim \mathbf{J}_7 = 52$ and hence $\mathbf{J}_7 = \mathbf{J}_8 = \mathbf{J}$.

When $K \neq \mathrm{GF}(2)$ we shall extend the bases obtained above with a sufficient number of new elements to obtain the bounds listed. (There is a slight abuse of notation here: the base elements above live in a vector space over GF(2) instead of over K, hence we first need to 'lift' them to the correct vector space. This is can be done in the obvious way, as all coefficients are either 0 or 1.)

Consider the case of $\mathbf{J}_2$. For $k \in K_\sigma^{(2)}$ we find

$$P(k,0) = E_{0101} + k_1^\sigma E_{00\bar{1}1} + k_0^\sigma E_{0011} + N(k)^\sigma E_{0\bar{1}01} + T(k) E_{0001}$$

and hence, after reduction by the basis elements of $\mathbf{J}_2$ we have obtained earlier, $(k_0^\sigma + k_1^\sigma + T(k))E_{\mathtt{0001}} \in \mathbf{J}_2$. Choosing $k_0 = 0$ and k_1 such that $k_1 + k_1^\sigma \neq 0$, we find that $E_{\mathtt{0001}} \in \mathbf{J}_2$, and this element may serve as the extra basis element we require to make $d_2 = 5$.

For $\mathbf{J}_3$ we need three extra elements. These arise from the equality

$$P(a, 0^2) = E_{\mathtt{0\bar{1}\bar{1}0}} + a^\sigma E_{\mathtt{10\bar{1}0}} + a^2 E_{\mathtt{\bar{1}001}} + a^{2+\sigma} E_{\mathtt{0101}} + aE_{\mathtt{---+}} + a^{1+\sigma} E_{\mathtt{++-+}}.$$

This proves that $a^\sigma E_{\mathtt{10\bar{1}0}} + a^2 E_{\mathtt{\bar{1}001}} + aE_{\mathtt{---+}} + a^{1+\sigma} E_{\mathtt{++-+}} \in \mathbf{J}_3$ for all $a \in K$. The case $a = 1$ yields an element which we had obtained before, but Lemma 3 proves that we may still find a further three linearly independent elements of this form, proving that $d_3 = 13$.

For $\mathbf{J}_4$ we first use $P(a, 0^2)$ of the previous case to 'split' $E_{\mathtt{10\bar{1}0}} + E_{\mathtt{\bar{1}001}}$ into separate basis vectors $E_{\mathtt{1010}}$ and $E_{\mathtt{\bar{1}001}}$. Second, we have

$$\begin{aligned} P(k, 0^3) = {} & E_{\mathtt{0110}} + k_0^2 E_{\mathtt{01\bar{1}0}} + T(k)^\sigma H_2 + k_1^2 E_{\mathtt{0\bar{1}10}} + N(k)^2 E_{\mathtt{0\bar{1}\bar{1}0}} \\ & + k_0 E_{\mathtt{0100}} + k_1 E_{\mathtt{0010}} + N(k) H_3 + k_0 N(k) E_{\mathtt{00\bar{1}0}} + k_1 N(k) E_{\mathtt{0\bar{1}00}} \end{aligned} \tag{7}$$

which reduces to

$$\begin{aligned} & k_0^2 E_{\mathtt{01\bar{1}0}} + T(k)^\sigma H_2 + k_1^2 E_{\mathtt{0\bar{1}10}} + k_0 E_{\mathtt{0100}} + k_1 E_{\mathtt{0010}} + k_0 N(k) E_{\mathtt{00\bar{1}0}} \\ & \quad + k_1 N(k) E_{\mathtt{0\bar{1}00}}. \end{aligned} \tag{8}$$

The special case $k_0 = 0$ yields elements of the form

$$k_1^\sigma H_2 + k_1^2 E_{\mathtt{0\bar{1}10}} + k_1 E_{\mathtt{0010}} + k_1^{1+\sigma} E_{\mathtt{0\bar{1}00}},$$

and hence, applying Lemma 3 in a similar way as before, we may split $E_{\mathtt{0\bar{1}10}} + E_{\mathtt{0\bar{1}00}} + E_{\mathtt{0010}} + H_2$ into four separate terms, giving three new basis elements.

Similarly, setting $k_1 = 0$ in (8) yields elements of the form

$$k_0^2 E_{\mathtt{01\bar{1}0}} + k_0^{2+\sigma} H_2 + k_0 E_{\mathtt{0100}} + k_0^{3+\sigma} E_{\mathtt{00\bar{1}0}}.$$

By setting $k_0 = 1$, $k_0 = a$ and $k_0 = a + 1$ with $a \neq 0, 1$ in this result, we obtain three linearly independent elements. This provides yet another two new basis elements, bringing the total up to 30, which proves that $d_4 = 29$.

We leave it to the reader to prove the cases $d_5 = 43$ and $d_6 = 60$ in a similar way. □

6 The Suzuki suboctagon

The set of all points with Van Maldeghem coordinates $(\vec{x})$ for which all $K_\sigma^{(2)}$-entries are zero (i.e., $k = k' = k'' = l = l' = l'' = 0$), is a *suboctagon* of $\mathsf{O}(K, \sigma)$. This is a so-called *half-thin* generalized octagon, a degenerate case with only two lines through each point. A half-thin octagon of this type can be constructed by taking as points the flags of any generalized quadrangle, and as lines the points and lines of that quadrangle. We call this particular suboctagon of $\mathsf{O}(K, \sigma)$ a *Suzuki* suboctagon because the corresponding quadrangle is known as a Suzuki quadrangle (a self polar Moufang quadrangle of indifferent type).

A Suzuki quadrangle can be described fairly easily in terms of the better known *symplectic* quadrangle $W(K)$, which is defined as follows:

- The points of $W(K)$ are the points of the 3-dimensionale projective space over K. We shall represent a point x by its coordinates (x_0, x_1, x_2, x_3).
- The lines of $W(K)$ are those lines of the 3-dimensional projective space whose *Plücker coordinates* satisfy $p_{01} = p_{23}$.

Recall that the Plücker coordinates of a line xx' of the projective 3-space are the six values $p_{ij} = -p_{ji} = \begin{vmatrix} x_i & x_j \\ x'_i & x'_j \end{vmatrix}$, with $0 \le i < j \le 3$. They satisfy the identity $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$ and the fact that a point and a line are incident can be expressed as the set of four equations $x_i p_{jk} + x_j p_{ki} + x_k p_{ij} = 0$, for all i, j, k such that $0 \le i < j < k \le 3$.

The Suzuki quadrangle $W(K, \sigma)$ is the following subgeometry of $W(K)$:

- The points of $W(K, \sigma)$ are those points x of $W(K)$ for which $x_0x_1 + x_2x_3 \in K^\sigma$. We shall write x_4 for the unique element of K that satisfies $x_4^\sigma = x_0x_1 + x_2x_3$.
- The lines of $W(K, \sigma)$ are those lines of $W(K)$ whose Plücker coordinates satisfy $p_{02}, p_{13}, p_{03}, p_{12} \in K^\sigma$. Each line of $W(K, \sigma)$ can therefore be characterized by a unique quadruple (y_0, y_1, y_2, y_3) such that $y_0^\sigma = p_{12}, y_1^\sigma = p_{03}, y_2^\sigma = p_{02}$ and $y_3^\sigma = p_{13}$.

(Hence $W(K, \sigma) = W(K)$ when K is a perfect field.)

Taking $y_4 = p_{01} = p_{23}$ we see that there is also a unique $y_4 \in K$ such that $y_4^\sigma = y_0y_1 + y_2y_3$. (Hence, the quadruples $(x_0, \dots, x_3)$ and $(y_0, \dots, y_3)$ satisfy the same conditions. In fact, interchanging x_i and y_i defines a polarity on $W(K, \sigma)$.)

The equations for point-line incidence can now be rewritten as

$$\begin{array}{lllll} x_0y_0^\sigma & + x_1y_2^\sigma & + x_2y_4 & & = 0, \\ x_0y_3^\sigma & + x_1y_1^\sigma & & + x_3y_4 & = 0, \\ x_0y_4 & & + x_2y_1^\sigma & + x_3y_2^\sigma & = 0, \\ & x_1y_4 & + x_2y_3^\sigma & + x_3y_0^\sigma & = 0. \end{array} \tag{9}$$

In [2] we have shown that every point Ke of the Suzuki suboctagon in the embedding $\mathcal{O}$ can be represented in the following way:

$$\begin{aligned} e = {} & x_0y_0e_{-+-+} + x_0y_1e_{+--+} + x_0y_2e_{0001} + x_0y_3e_{00\bar{1}0} \\ & + x_1y_0e_{-++-} + x_1y_1e_{+-+-} + x_1y_2e_{0010} + x_1y_3e_{000\bar{1}} \\ & + x_2y_0e_{0100} + x_2y_1e_{1000} + x_2y_2e_{++++} + x_2y_3e_{++--} \\ & + x_3y_0e_{\bar{1}000} + x_3y_1e_{0\bar{1}00} + x_3y_2e_{--++} + x_3y_3e_{----}, \end{aligned} \tag{10}$$

with $x_0, \dots, x_3, y_0, \dots, y_3 \in K$. In other words, e can be expressed as a tensor product of two vectors $x = (x_0, \dots, x_3)$ and $y = (y_0, \dots, y_3)$ from a 4-dimensional vector space over K.

Moreover, not surprisingly, each such pair $(x_0, \dots, x_3)$ and $(y_0, \dots, y_3)$ can be shown to satisfy $x_0x_1 + x_2x_3, y_0y_1 + y_2y_3 \in K^\sigma$ and the set of equations (9), with the same definitions for x_4 and y_4. In other words, x and y represent an incident point-line pair of $W(K, \sigma)$.

Now, applying (2) to (10), we obtain

$$\begin{aligned}Q(e)^{\sigma/2} = {} & x_0^\sigma y_0^\sigma E_{1010} + x_0^\sigma y_1^\sigma E_{0\bar{1}01} + x_0^\sigma y_2^\sigma E_{1001} + x_0^\sigma y_3^\sigma E_{0\bar{1}10} + x_0^\sigma y_4 E_{+-++}\\ & + x_1^\sigma y_0^\sigma E_{010\bar{1}} + x_1^\sigma y_1^\sigma E_{\bar{1}0\bar{1}0} + x_1^\sigma y_2^\sigma E_{01\bar{1}0} + x_1^\sigma y_3^\sigma E_{\bar{1}00\bar{1}} + x_1^\sigma y_4 E_{-+--}\\ & + x_2^\sigma y_0^\sigma E_{0110} + x_2^\sigma y_1^\sigma E_{\bar{1}001} + x_2^\sigma y_2^\sigma E_{0101} + x_2^\sigma y_3^\sigma E_{\bar{1}010} + x_2^\sigma y_4 E_{-+++}\\ & + x_3^\sigma y_0^\sigma E_{100\bar{1}} + x_3^\sigma y_1^\sigma E_{0\bar{1}\bar{1}0} + x_3^\sigma y_2^\sigma E_{10\bar{1}0} + x_3^\sigma y_3^\sigma E_{0\bar{1}0\bar{1}} + x_3^\sigma y_4 E_{+---}\\ & + x_4 y_0^\sigma E_{+++-} + x_4 y_1^\sigma E_{---+} + x_4 y_2^\sigma E_{++-+} + x_4 y_3^\sigma E_{--+-} + x_4 y_4 H_4.\end{aligned} \tag{11}$$

(Although the left hand side contains $\sigma/2$, the fact that the right hand side does not, proves that this formula remains well-defined also when K is not a perfect field.)

Hence $Q(e)^{\sigma/2}$ is a tensor product of the vectors $(x_0^\sigma, \ldots, x_3^\sigma, x_4)$ and $(y_0^\sigma, \ldots, y_3^\sigma, y_4)$. (This tensor product can also be expressed as the restriction of operator $*$ of [1,3] to two 5-dimensional subspaces of $\mathbf{W}$.)

Theorem 5 *The (projective) dimension of the subspace of* $\mathcal{O}'$ *spanned by the points of a Suzuki suboctagon is* 24 *when* $|K| > 2$ *and* 15 *when* $K = \mathrm{GF}(2)$.

Proof Write $\mathbf{S}$ for the subspace of $\mathbf{J}$ generated by all $E \in \mathbf{J}$ such that KE belongs to the Suzuki suboctagon. We must prove that $\dim \mathbf{S} = 16$ when $K = \mathrm{GF}(2)$ and $\dim \mathbf{S} = 25$ otherwise. It follows immediately from (11) that $\dim \mathbf{S} \le 25$.

We first consider the case $K = \mathrm{GF}(2)$, where σ is the identity. In that case (9) shows that the values of $x_i y_4$ can always be expressed as sums of terms $x_j y_k$ with $j, k \ne 4$. Now, multiplying the first row of (10) by x_1, the last row by x_2 and adding, yields $x_4^\sigma y_0^\sigma + x_1^2 y_2^\sigma + x_2^2 y_3^\sigma = 0$, and hence $x_4 y_0 = x_1^\sigma y_2 + x_2^\sigma y_3$, proving that also $x_4 y_0$ can be expressed as sums of terms $x_j y_k$ with $j, k \ne 4$. Similar identities can be found for $x_4 y_1, \ldots, x_4 y_3$. Finally, we claim that $x_0 y_4 + \cdots + x_4 y_4 = 0$ when $K = \mathrm{GF}(2)$. This is most easily verified by checking that the sum of the coefficients of E_{+-++}, E_{-+--}, E_{-+++}, E_{+---} and H_4 in the tables of Appendix A is indeed zero whenever $k = k' = k'' = l = l' = l'' = 0$.

These observations prove that when $K = \mathrm{GF}(2)$ the nine coefficients in (11) that involve x_4 or y_4 are linear combinations of the others. It follows that $\dim \mathbf{S}$ can be at most 16.

To prove equality it is sufficient to establish a basis of $\mathbf{S}$ of size 16. We shall make use of the fact that the Suzuki suboctagon is left invariant by the automorphism w.

The first eight basis vectors are taken as subsequent images $P, P \cdot w, \ldots P \cdot w^7$, i.e.,

$$E_{1001},\ E_{0\bar{1}01},\ E_{0\bar{1}\bar{1}0},\ E_{0\bar{1}0\bar{1}},\ E_{\bar{1}00\bar{1}},\ E_{010\bar{1}},\ E_{0110},\ E_{0101}.$$

Now, consider

$$\begin{aligned}P(1,0,1,0^2) + P(1,0^4) + P(0^2,1,0^2) = {} & E_{0101} + E_{0\bar{1}0\bar{1}} + E_{1010} + E_{-+++}\\ & + E_{++-+} + H_4.\end{aligned}$$

Reducing this element with respect to the first eight basis elements, provides us with another eight:

$$E_{1010} + E_{-+++} + E_{++-+} + H_4,\ E_{\bar{1}001} + E_{++-+} + E_{+-++} + H_4,$$
$$E_{10\bar{1}0} + E_{+-++} + E_{---+} + H_4,\ E_{01\bar{1}0} + E_{---+} + E_{+---} + H_4,$$
$$E_{\bar{1}0\bar{1}0} + E_{+---} + E_{--+-} + H_4,\ E_{100\bar{1}} + E_{--+-} + E_{-+--} + H_4,$$
$$E_{\bar{1}010} + E_{-+--} + E_{+++-} + H_4,\ E_{0\bar{1}10} + E_{+++-} + E_{-+++} + H_4,$$

This proves that $\dim \mathbf{S} = 16$ when $K = \mathrm{GF}(2)$.

We may generalize this to the case $K \neq \mathrm{GF}(2)$ as follows. Let $a \in K$. We have

$$P(a, 0, 1, 0^2) + P(a, 0^4) + P(0^2, 1, 0^2) = a^2 E_{0101} + E_{0\bar{1}0\bar{1}} + a^\sigma E_{1010}$$
$$+ a^2 E_{-+++} + aE_{++-+} + aH_4.$$

Reducing this as before, we obtain for each $a \in K$ the following element of $\mathbf{S}$

$$V(a) \stackrel{\text{def}}{=} a^\sigma E_{1010} + a^2 E_{-+++} + a(E_{++-+} + H_4),$$

and by Lemma 3 we may find $a, b, c \in K$ such that $V(a)$, $V(b)$, and $V(c)$ are linearly independent. It follows that E_{1010}, E_{-+++}, and $E_{++-+} + H_4$ belong to $\mathbf{S}$. Repeatedly applying w to these elements, we see that each canonical basis element of $\mathbf{J}$ that occurs in (11) also belongs to $\mathbf{S}$. Hence $\dim \mathbf{S} = 25$. □

A computer result of [5, Section 8.7.1] states that the universal embedding of $(2W(2))^D$ has dimension 15. This clarifies why in the theorem above the dimension for the case $K = \mathrm{GF}(2)$ goes down to this value.

Finally, comparing (10) with the 'top left' part of (11), we observe that the same projection that maps the embedding $\mathcal{O}'(K)$ onto $\mathcal{O}(K^\sigma)$ also maps the embedding of the Suzuki suboctagon of Theorem 5 onto the 15-dimensional embedding of [2] (with base field K^σ instead of K).

Acknowledgements I would like to thank the anonymous referees for many useful comments and suggestions.

Appendix A: Tables

The formulas below express the elements $P(\cdots)$ of $\mathbf{J}$ in terms of the canonical basis elements of $\mathbf{J}$. They are arranged in eight separate tables, corresponding to arities 0 upto 7. For typographical reasons the formulas are given as tables of coordinates. The expressions in the right hand column are the coordinates for the canonical basis vectors of $\mathbf{J}$ indicated in the left hand column. We refer to the basis vectors E_r and $H_1 \dots H_4$ by the symbols r and $1 \dots 4$. For example, the table for arity 2 should be interpreted as

$$P(k, b) = E_{0101} + k_1^\sigma E_{00\bar{1}1} + k_0^\sigma E_{0011} + N(k)^\sigma E_{0\bar{1}01} + b^\sigma E_{1001}$$
$$+ T(k) E_{0001}.$$

For arity 2 and above, each table consists of two parts. The first part contains the coordinates that correspond to the long roots of Φ, the second part corresponds to the short roots. As was explained in Sect. 3 the first part of each table provides a 25-dimensional embedding of $\mathsf{O}(K,\sigma)$ which is essentially the same as the embedding $\mathcal{O}$ of [2], up to an application of σ and a renaming of the basis vectors.

Arity 0. Value of P.

`1001`	1

Arity 1. Values of $P(a)$ with $a \in K$.

$0\bar{1}01$	1
1001	a^σ

Arity 2. Values of $P(k,b)$ with $b \in K$, $k \in K_\sigma^{(2)}$.

0101	1
$00\bar{1}1$	k_1^σ
0011	k_0^σ
$0\bar{1}01$	$N(k)^\sigma$
1001	b^σ
0001	$T(k)$

Arity 3. Values of $P(a,l,a')$ with $a, a' \in K$, $l \in K_\sigma^{(2)}$.

$0\bar{1}\bar{1}0$	1
$10\bar{1}0$	a^σ
$1\bar{1}00$	l_0^σ
$\bar{1}001$	a^2
0101	$a^{2+\sigma}$
$00\bar{1}1$	l_1^σ
0011	$a^2 l_0^\sigma$
$0\bar{1}01$	a'^σ
1001	$N(l)^\sigma + a^\sigma a'^\sigma$
$---+$	a
$+-++$	$a l_0^\sigma$
$++-+$	$a^{1+\sigma}$
$+--+$	$T(l)$
0001	$aT(l)$

Arity 4. Values of $P(k, b, k', b')$ with $b, b' \in K$, $k, k' \in K_\sigma^{(2)}$.

0110	1
$01\bar{1}0$	k_0^2
2	$T(k)^\sigma$
$0\bar{1}10$	k_1^2
1010	$b^\sigma + k_0^\sigma k_0'^\sigma$
1100	$k_0'^\sigma$
$0\bar{1}\bar{1}0$	$N(k)^2$
$10\bar{1}0$	$b^\sigma k_0^2 + k_1^\sigma k_0'^\sigma$
$1\bar{1}00$	$b^\sigma T(k)^\sigma + N(k)^\sigma k_0'^\sigma$
$\bar{1}001$	b^2
0101	b'^σ
$00\bar{1}1$	$b^2 k_0'^\sigma + k_0^2 k_1'^\sigma + b'^\sigma k_1^\sigma$
0011	$k_1'^\sigma + b'^\sigma k_0^\sigma$
$0\bar{1}01$	$b^{2+\sigma} + b^2 k_0^\sigma k_0'^\sigma + T(k)^\sigma k_1'^\sigma + b'^\sigma N(k)^\sigma$
1001	$N(k')^\sigma + b^\sigma b'^\sigma$

0100	k_0
0010	k_1
3	$N(k)$
$00\bar{1}0$	$k_0 N(k)$
-+-+	bk_0
-+++	b
$0\bar{1}00$	$k_1 N(k)$
--++	bk_1
---+	$bN(k)$
1000	$b^\sigma k_0 + T(k) k_0'^\sigma$
++++	$T(k')$
+-++	$bk_0^\sigma k_0'^\sigma + b^{1+\sigma} + k_1 T(k')$
++-+	$bk_0'^\sigma + k_0 T(k')$
+--+	$bT(k)k_0'^\sigma + b^{1+\sigma} k_0 + N(k)T(k')$
0001	$k_0 k_1'^\sigma + b'^\sigma T(k) + bT(k')$

Arity 5. Values of $P(a, l, a', l', a'')$ with $a, a', a'' \in K$, $l, l' \in K_\sigma^{(2)}$.

$0\bar{1}0\bar{1}$	1
$\bar{1}0\bar{1}0$	l_0^2
1	$a^\sigma l_0'^\sigma + T(l)^\sigma$
$\bar{1}\bar{1}00$	$l_0'^\sigma$
$100\bar{1}$	a^σ

$\bar{1}010$	a^2
0110	$a^{2+\sigma}$
$01\bar{1}0$	$a^\sigma l_0^2$
2	$T(l)^\sigma$
$0\bar{1}10$	$a'^\sigma + l_0^\sigma l_0'^\sigma$
1010	$a^\sigma a'^\sigma + a^\sigma l_0^\sigma l_0'^\sigma + l_1^2$
1100	$a^\sigma T(l)^\sigma + a^{2\sigma} l_0'^\sigma$
$0\bar{1}\bar{1}0$	a''^σ
$10\bar{1}0$	$a'^2 + a^\sigma a''^\sigma$
$1\bar{1}00$	$l_1'^\sigma + a''^\sigma l_0^\sigma$
$\bar{1}001$	$a'^\sigma l_0^2 + l_1^\sigma l_0'^\sigma + a^2 a''^\sigma$
0101	$N(l)^2 + a^2 a'^2 + a^\sigma a'^\sigma l_0^2 + a^\sigma l_1^\sigma l_0'^\sigma + a^{2+\sigma} a''^\sigma$
$00\bar{1}1$	$a'^2 l_0'^\sigma + l_0^2 l_1'^\sigma + a''^\sigma l_1^\sigma$
0011	$a'^\sigma T(l)^\sigma + N(l)^\sigma l_0'^\sigma + a^2 l_1'^\sigma + a^2 a''^\sigma l_0^\sigma$
$0\bar{1}01$	$N(l')^\sigma + a'^\sigma a''^\sigma$
1001	$a'^{2+\sigma} + a'^2 l_0^\sigma l_0'^\sigma + T(l)^\sigma l_1'^\sigma + a^\sigma N(l')^\sigma + [N(l)^\sigma + a^\sigma a'^\sigma] a''^\sigma$

`--+-`	a
`+++-`	$a^{1+\sigma}$
`----`	l_0
`+-+-`	l_1
4	$aa' + N(l)$
`+---`	a'
$\bar{1}000$	al_0
`++--`	$a^\sigma l_0$
0100	$a^{1+\sigma} l_0$
0010	al_1
3	$a^\sigma l_0'^\sigma + l_0 l_1$
$00\bar{1}0$	$a' l_0$
`-+-+`	$aa' l_0 + l_0 N(l)$
`-+++`	$a^2 a' + aN(l)$
$0\bar{1}00$	$T(l')$
`--++`	$a'^\sigma l_0 + T(l) l_0'^\sigma + aT(l')$
`---+`	$a' l_0'^\sigma + aa''^\sigma + l_0 T(l')$
1000	$a' l_1 + a^\sigma T(l')$
`++++`	$aa' l_1 + a^\sigma a'^\sigma l_0 + a^\sigma T(l) l_0'^\sigma + l_1 N(l) + a^{1+\sigma} T(l')$
`+-++`	$a' l_0{}^\sigma l_0'^\sigma + a l_1'^\sigma + a'^{1+\sigma} + aa''^\sigma l_0{}^\sigma + l_1 T(l')$
`++-+`	$a^\sigma a' l_0'^\sigma + a' N(l) + aa'^2 + a^{1+\sigma} a''^\sigma + a^\sigma l_0 T(l')$
`+--+`	$l_0 l_1'^\sigma + a''^\sigma T(l) + a' T(l')$
0001	$a l_0 l_1'^\sigma + a' T(l) l_0'^\sigma + a'^{1+\sigma} l_0 + aa''^\sigma T(l) + T(l')[aa' + N(l)]$

Arity 6. Values of $P(k, b, k', b', k'', b'')$ with $b, b', b'' \in K$, $k, k', k'' \in K_\sigma^{(2)}$.

$010\bar{1}$	1
$001\bar{1}$	k_0^σ
$00\bar{1}\bar{1}$	k_1^σ
$\bar{1}100$	$k_0''^\sigma$
$0\bar{1}0\bar{1}$	$N(k)^\sigma$
$\bar{1}0\bar{1}0$	$b^2 + k_0^2 k_0'^2 + k_1^\sigma k_0''^\sigma$
1	$b^\sigma k_0''^\sigma + T(k')^\sigma$
$\bar{1}\bar{1}00$	$T(k)^\sigma k_0'^2 + N(k)^\sigma k_0''^\sigma + b^2 k_0^\sigma$
$100\bar{1}$	b^σ
$\bar{1}010$	$k_0'^2 + k_0^\sigma k_0''^\sigma$
0110	b''^σ
$01\bar{1}0$	$b'^\sigma + k_0'^\sigma k_0''^\sigma + b''^\sigma k_0^2$
2	$T(k')^\sigma + b'^\sigma k_0^\sigma + k_0^\sigma k_0'^\sigma k_0''^\sigma + b''^\sigma T(k)^\sigma$
$0\bar{1}10$	$b^\sigma k_0'^2 + b^\sigma k_0^\sigma k_0''^\sigma + k_0^\sigma T(k')^\sigma + b'^\sigma k_0^{2\sigma} + k_0^{2\sigma} k_0'^\sigma k_0''^\sigma + b''^\sigma k_1^2$
1010	$b'^2 + k_0^\sigma k_1''^\sigma + [b^\sigma + k_0^\sigma k_0'^\sigma] b''^\sigma$
1100	$k_1''^\sigma + b''^\sigma k_0'^\sigma$
$0\bar{1}\bar{1}0$	$b^{2+\sigma} + b^\sigma k_1^\sigma k_0''^\sigma + k_1^\sigma T(k')^\sigma + b^\sigma k_0^2 k_0'^2 + b'^\sigma T(k)^2 + T(k)^2 k_0'^\sigma k_0''^\sigma$ $+ b''^\sigma N(k)^2$
$10\bar{1}0$	$b^\sigma b'^\sigma + b^\sigma k_0'^\sigma k_0''^\sigma + k_1'^2 + b'^2 k_0^2 + k_1^\sigma k_1''^\sigma + [b^\sigma k_0^2 + k_1^\sigma k_0'^\sigma] b''^\sigma$
$1\bar{1}00$	$b^\sigma T(k')^\sigma + b^{2\sigma} k_0''^\sigma + b^\sigma b'^\sigma k_0^\sigma + b^\sigma k_0^\sigma k_0'^\sigma k_0''^\sigma + k_0^\sigma k_1'^2 + b'^2 T(k)^\sigma$ $+ N(k)^\sigma k_1''^\sigma + [b^\sigma T(k)^\sigma + N(k)^\sigma k_0'^\sigma] b''^\sigma$
$\bar{1}001$	$b'^\sigma k_0'^2 + k_1'^\sigma k_0''^\sigma + b^2 b''^\sigma$
0101	$N(k'')^\sigma + b'^\sigma b''^\sigma$
$00\bar{1}1$	$b'^\sigma T(k')^\sigma + N(k')^\sigma k_0''^\sigma + b^2 k_1''^\sigma + b'^2 k_0^2 k_0''^\sigma + k_0^2 k_0'^2 k_1''^\sigma + k_1^\sigma N(k'')^\sigma$ $+ [b^2 k_0'^\sigma + k_0^2 k_1'^\sigma + b'^\sigma k_1^\sigma] b''^\sigma$
0011	$b'^2 k_0''^\sigma + k_0'^2 k_1''^\sigma + k_0^\sigma N(k'')^\sigma + [k_1'^\sigma + b'^\sigma k_0^\sigma] b''^\sigma$
$0\bar{1}01$	$N(k')^2 + b^2 b'^2 + b^\sigma b'^\sigma k_0'^2 + b^\sigma k_1'^\sigma k_0''^\sigma + b'^2 T(k)^\sigma k_0''^\sigma + T(k)^\sigma k_0'^2 k_1''^\sigma$ $+ b'^\sigma k_0^\sigma T(k')^\sigma + k_0^\sigma N(k')^\sigma k_0''^\sigma + b^2 k_0^\sigma k_1''^\sigma + N(k)^\sigma N(k'')^\sigma$ $+ [b^{2+\sigma} + b^2 k_0^\sigma k_0'^\sigma + T(k)^\sigma k_1'^\sigma + b'^\sigma N(k)^\sigma] b''^\sigma$
1001	$b'^{2+\sigma} + b'^2 k_0'^\sigma k_0''^\sigma + T(k')^\sigma k_1''^\sigma + b^\sigma N(k'')^\sigma + [N(k')^\sigma + b^\sigma b'^\sigma] b''^\sigma$
$000\bar{1}$	$T(k)$
−++−	k_0'
−+−−	$k_0 k_0' + b$
−−+−	$b k_0^\sigma + k_1 k_0'$
+++−	b'
−−−−	$N(k) k_0' + b T(k)$

+-+-	$b^\sigma k_0' + b'k_1 + k_0^\sigma k_1'$
4	$bb' + N(k')$
+---	$b^\sigma k_0 k_0' + b'N(k) + T(k)k_1' + b^{1+\sigma}$
$\bar{1}000$	$bk_0' + k_0 k_0'^2 + T(k)k_0''^\sigma$
++--	$b'k_0 + k_1'$
0100	$b''^\sigma k_0 + T(k'')$
0010	$b'k_0' + b''^\sigma k_1 + k_0^\sigma T(k'')$
3	$b'k_0 k_0' + N(k') + b''^\sigma N(k) + T(k)T(k'')$
$00\bar{1}0$	$bb'k_0 + b'k_0^2 k_0' + k_0 k_0' k_1' + bk_1' + T(k)k_0'^\sigma k_0''^\sigma + b'^\sigma T(k) + b''^\sigma k_0 N(k) + k_1^\sigma T(k'')$
-+-+	$b'k_0 k_0''^\sigma + b'^\sigma k_0' + T(k')k_0''^\sigma + bb''^\sigma k_0 + T(k'')[b + k_0 k_0']$
-+++	$b'k_0''^\sigma + bb''^\sigma + k_0' T(k'')$
$0\bar{1}00$	$bb'k_1 + b^\sigma k_0 k_0'^2 + bk_0{}^\sigma k_1' + k_1 k_0' k_1' + b^{1+\sigma} k_0' + k_0{}^\sigma T(k)k_0'^\sigma k_0''^\sigma + b^\sigma T(k)k_0''^\sigma + b'^\sigma k_0{}^\sigma T(k) + b'T(k)^\sigma k_0' + T(k)T(k')^\sigma + b''^\sigma k_1 N(k) + N(k)^\sigma T(k'')$
--++	$bb'k_0' + b'^\sigma k_0{}^\sigma k_0' + b'k_1 k_0''^\sigma + k_0{}^\sigma T(k')k_0''^\sigma + k_0' N(k') + bb''^\sigma k_1 + T(k'')[bk_0^\sigma + k_1 k_0']$
---+	$bb'k_0 k_0' + b^2 b' + b'^\sigma T(k)k_0' + T(k)T(k')k_0''^\sigma + k_0 k_0' N(k') + b'N(k)k_0''^\sigma + bN(k') + bb''^\sigma N(k) + T(k'')[bT(k) + N(k)k_0']$
1000	$b'^2 k_0 + b'k_1' + T(k)k_1''^\sigma + b''^\sigma [b^\sigma k_0 + T(k)k_0'^\sigma] + b^\sigma T(k'')$
++++	$k_0' k_1''^\sigma + b''^\sigma T(k') + b'T(k'')$
+-++	$b'k_0{}^\sigma k_0'^\sigma k_0''^\sigma + b^\sigma b'k_0''^\sigma + bk_0{}^\sigma k_1''^\sigma + k_1 k_0' k_1''^\sigma + bb'^2 + b'^{1+\sigma} k_0{}^\sigma + b'N(k') + b''^\sigma [bk_0{}^\sigma k_0'^\sigma + b^{1+\sigma} + k_1 T(k')] + T(k'')[b^\sigma k_0' + k_0^\sigma k_1' + b'k_1]$
++-+	$b'k_0'^\sigma k_0''^\sigma + k_0 k_0' k_1''^\sigma + bk_1''^\sigma + b'^{1+\sigma} + b''^\sigma [bk_0'^\sigma + k_0 T(k')] + T(k'')[k_1' + b'k_0])$
+--+	$b^\sigma b'k_0 k_0''^\sigma + bb'^2 k_0 + b^\sigma b'^\sigma k_0' + bb'k_1' + b'k_0 N(k') + b'T(k)k_0'^\sigma k_0''^\sigma + N(k)k_0' k_1''^\sigma + bT(k)k_1''^\sigma + b^\sigma T(k')k_0''^\sigma + b'^{1+\sigma} T(k) + k_1' N(k') + b''^\sigma [bT(k)k_0'^\sigma + b^{1+\sigma} k_0 + N(k)T(k')] + T(k'')[b^{1+\sigma} + b^\sigma k_0 k_0' + T(k)k_1' + b'N(k)]$
0001	$bk_0' k_1''^\sigma + b'^2 k_0 k_0''^\sigma + k_0 k_0'^2 k_1''^\sigma + b'^{1+\sigma} k_0' + b'T(k')k_0''^\sigma + T(k)N(k'')^\sigma + b''^\sigma [k_0 k_1'^\sigma + b'^\sigma T(k) + bT(k')] + T(k'')[N(k') + bb']$

Arity 7. Values of $P(a, l, a', l', a'', l'', a''')$ with $a, a', a'', a''' \in K, l, l', l'' \in K_\sigma^{(2)}$.

$\bar{1}00\bar{1}$	1
$010\bar{1}$	a^σ
$001\bar{1}$	l_0^σ
$00\bar{1}\bar{1}$	$l_0''^\sigma$
$\bar{1}100$	$l_1^\sigma + a^2 l_0''^\sigma$
$0\bar{1}0\bar{1}$	a'''^σ

$\bar{1}0\bar{1}0$	$a''^\sigma + l_0'^\sigma l_0''^\sigma + a'''^\sigma l_0^2$
1	$a''^\sigma l_0^\sigma + l_0^\sigma l_0'^\sigma l_0''^\sigma + a'^\sigma l_0''^\sigma + a^\sigma l_1''^\sigma + [a^\sigma l_0'^\sigma + T(l)^\sigma]a'''^\sigma$
$\bar{1}\bar{1}00$	$l_1''^\sigma + a'''^\sigma l_0'^\sigma$
$100\bar{1}$	$l_0'^2 + l_0^\sigma l_0''^\sigma + a^\sigma a'''^\sigma$
$\bar{1}010$	$a'^\sigma + a^2 a'''^\sigma$
0110	$N(l)^\sigma + a^2 l_0'^2 + a^2 l_0^\sigma l_0''^\sigma + a^\sigma a'^\sigma + a^{2+\sigma} a'''^\sigma$
$01\bar{1}0$	$a'^2 + l_0^2 l_0'^2 + l_1^\sigma l_0''^\sigma + a^\sigma a''^\sigma + a^\sigma l_0'^\sigma l_0''^\sigma + a^\sigma a'''^\sigma l_0^2$
2	$T(l')^\sigma + a''^\sigma l_0^\sigma + l_0^\sigma l_0'^\sigma l_0''^\sigma + a'''^\sigma T(l)^\sigma$
$0\bar{1}10$	$a''^2 + l_0^\sigma l_1''^\sigma + [a'^\sigma + l_0^\sigma l_0'^\sigma]a'''^\sigma$
1010	$a'^\sigma l_0'^2 + a'^\sigma l_0^\sigma l_0''^\sigma + l_0^\sigma T(l')^\sigma + a''^\sigma l_0^{2\sigma} + l_0^{2\sigma} l_0'^\sigma l_0''^\sigma + a^\sigma a''^2 + a^\sigma l_0^\sigma l_1''^\sigma + [a^\sigma a'^\sigma + a^\sigma l_0^\sigma l_0'^\sigma + l_1^2]a'''^\sigma$
1100	$T(l)^\sigma l_0'^2 + N(l)^\sigma l_0''^\sigma + a'^2 l_0^\sigma + a^{2\sigma} l_1''^\sigma + a^\sigma T(l')^\sigma + a^\sigma a''^\sigma l_0^\sigma + a^\sigma l_0^\sigma l_0'^\sigma l_0''^\sigma + [a^\sigma T(l)^\sigma + a^{2\sigma} l_0'^\sigma]a'''^\sigma$
$0\bar{1}\bar{1}0$	$N(l'')^\sigma + a''^\sigma a'''^\sigma$
$10\bar{1}0$	$a''^\sigma l_0'^2 + l_1'^\sigma l_0''^\sigma + a^\sigma N(l'')^\sigma + [a'^2 + a^\sigma a''^\sigma]a'''^\sigma$
$1\bar{1}00$	$a''^2 l_0''^\sigma + l_0'^2 l_1''^\sigma + l_0^\sigma N(l'')^\sigma + [l_1'^\sigma + a''^\sigma l_0^\sigma]a'''^\sigma$
$\bar{1}001$	$l_1'^2 + a'^\sigma a''^\sigma + a'^\sigma l_0'^\sigma l_0''^\sigma + a''^2 l_0^2 + l_1^\sigma l_1''^\sigma + a^2 N(l'')^\sigma + [a'^\sigma l_0^2 + l_1^\sigma l_0'^\sigma + a^2 a''^\sigma]a'''^\sigma$
0101	$a'^{2+\sigma} + a'^\sigma l_1^\sigma l_0''^\sigma + l_1^\sigma T(l')^\sigma + a'^\sigma l_0^2 l_0'^2 + a''^\sigma T(l)^2 + T(l)^2 l_0'^\sigma l_0''^\sigma + a^2 a''^\sigma l_0'^2 + a^2 l_1'^\sigma l_0''^\sigma + a^\sigma a'^\sigma a''^\sigma + a^\sigma a'^\sigma l_0'^\sigma l_0''^\sigma + a^\sigma l_1'^2 + a^\sigma a''^2 l_0^2 + a^\sigma l_1^\sigma l_1''^\sigma + a^{2+\sigma} N(l'')^\sigma + [N(l)^2 + a^2 a'^2 + a^\sigma a'^\sigma l_0^2 + a^\sigma l_1^\sigma l_0'^\sigma + a^{2+\sigma} a''^\sigma]a'''^\sigma$
$00\bar{1}1$	$a''^\sigma T(l')^\sigma + N(l')^\sigma l_0''^\sigma + a'^2 l_1''^\sigma + a''^2 l_0^2 l_0''^\sigma + l_0^2 l_0'^2 l_1''^\sigma + l_1^\sigma N(l'')^\sigma + [a'^2 l_0'^\sigma + l_0^2 l_1'^\sigma + a''^\sigma l_1^\sigma]a'''^\sigma$
0011	$a'^\sigma T(l')^\sigma + a'^{2\sigma} l_0''^\sigma + a'^\sigma a''^\sigma l_0^\sigma + a'^\sigma l_0^\sigma l_0'^\sigma l_0''^\sigma + l_0^\sigma l_1'^2 + a''^2 T(l)^\sigma + N(l)^\sigma l_1''^\sigma + a^2 a''^2 l_0''^\sigma + a^2 l_0'^2 l_1''^\sigma + a^2 l_0^\sigma N(l'')^\sigma + [a'^\sigma T(l)^\sigma + N(l)^\sigma l_0'^\sigma + a^2 l_1'^\sigma + a^2 a''^\sigma l_0^\sigma]a'''^\sigma$
$0\bar{1}01$	$a''^{2+\sigma} + a''^2 l_0'^\sigma l_0''^\sigma + T(l')^\sigma l_1''^\sigma + a'^\sigma N(l'')^\sigma + [N(l')^\sigma + a'^\sigma a''^\sigma]a'''^\sigma$
1001	$N(l')^2 + a'^2 a''^2 + a'^\sigma a''^\sigma l_0'^2 + a'^\sigma l_1'^\sigma l_0''^\sigma + a''^2 T(l)^\sigma l_0''^\sigma + T(l)^\sigma l_0'^2 l_1''^\sigma + a''^\sigma l_0^\sigma T(l')^\sigma + l_0^\sigma N(l')^\sigma l_0''^\sigma + a'^2 l_0^\sigma l_1''^\sigma + a^\sigma a''^{2+\sigma} + a^\sigma a''^2 l_0'^\sigma l_0''^\sigma + a^\sigma T(l')^\sigma l_1''^\sigma + [N(l)^\sigma + a^\sigma a'^\sigma]N(l'')^\sigma + [a'^{2+\sigma} + a'^2 l_0^\sigma l_0'^\sigma + T(l)^\sigma l_1'^\sigma + a^\sigma N(l')^\sigma + a''^\sigma N(l)^\sigma + a^\sigma a'^\sigma a''^\sigma]a'''^\sigma$
$000\bar{1}$	l_0'
−++−	$al_0' + T(l)$
−+−−	$al_0''^\sigma + l_0 l_0' + a'$
−−+−	$a'' + aa'''^\sigma$
+++−	$al_0{}^\sigma l_0''^\sigma + a^\sigma a'' + al_0'^2 + a' l_0{}^\sigma + l_1 l_0' + a^{1+\sigma} a'''^\sigma$
−−−−	$a'''^\sigma l_0 + T(l'')$
+−+−	$a'' l_0' + a'''^\sigma l_1 + l_0^\sigma T(l'')$

4	$aa''l_0''^\sigma + a''l_0l_0' + a'a'' + a'^\sigma l_0''^\sigma + T(l')^\sigma + a'''^\sigma[aa' + N(l)] + T(l'')[al_0' + T(l)]$
+---	$a''l_0''^\sigma + a'a'''^\sigma + l_0'T(l'')$
$\bar{1}000$	$a''l_0 + l_1' + aa'''^\sigma l_0 + aT(l'')$
++--	$a'l_0' + l_0l_0'^2 + T(l)l_0''^\sigma + a^\sigma a'''^\sigma l_0 + a^\sigma T(l'')$
0100	$aa'l_0' + a^\sigma a''l_0 + al_0l_0'^2 + aT(l)l_0''^\sigma + N(l)l_0' + a^\sigma l_1' + a'T(l) + a^{1+\sigma}a'''^\sigma l_0 + a^{1+\sigma}T(l'')$
0010	$aa''l_0' + a'^\sigma l_0' + a''l_1 + l_0^\sigma l_1' + aa'''^\sigma l_1 + al_0^\sigma T(l'')$
3	$a''l_0l_0' + l_0^\sigma l_0'^\sigma l_0''^\sigma + a^\sigma l_1''^\sigma + a''^\sigma l_0^\sigma + l_0'l_1' + a'''^\sigma[a^\sigma l_0'^\sigma + l_0l_1] + T(l)T(l'')$
$00\bar{1}0$	$a''l_0l_0''^\sigma + a''^\sigma l_0' + T(l')l_0''^\sigma + a'a'''^\sigma l_0 + T(l'')[a' + l_0l_0']$
-+-+	$aa''l_0l_0''^\sigma + aa''^\sigma l_0' + aT(l')l_0''^\sigma + a'a''l_0 + a''l_0^2l_0' + l_0l_0'l_1' + T(l)l_0'^\sigma l_0''^\sigma + a'l_1' + a''^\sigma T(l) + a'''^\sigma[aa'l_0 + l_0N(l)] + T(l'')[aa' + al_0l_0' + l_1^\sigma]$
-+++	$aa''l_0l_0' + aa'^\sigma l_0''^\sigma + a^2a''l_0''^\sigma + al_0'l_1' + a'^\sigma l_0l_0' + a''N(l) + T(l)l_1' + a'^{1+\sigma} + a'''^\sigma[a^2a' + aN(l)] + T(l'')[aT(l) + a^2l_0']$
$0\bar{1}00$	$l_0'l_1''^\sigma + a'''^\sigma T(l') + a''T(l'')$
--++	$al_0'l_1''^\sigma + a''^2l_0 + a''l_1' + T(l)l_1''^\sigma + a'''^\sigma[a'^\sigma l_0 + T(l)l_0'^\sigma + aT(l')] + T(l'')[a'^\sigma + aa'']$
---+	$aN(l'')^\sigma + a''l_0'^\sigma l_0''^\sigma + l_0l_0'l_1''^\sigma + a'l_1''^\sigma + a''^{1+\sigma} + a'''^\sigma[a'l_0'^\sigma + aa''^\sigma + l_0T(l')] + T(l'')[l_1' + a''l_0]$
1000	$a^\sigma l_0'l_1''^\sigma + a'a''l_0' + a''^\sigma l_0^\sigma l_0' + a''l_1l_0''^\sigma + l_0^\sigma T(l')l_0''^\sigma + l_0'N(l') + a'''^\sigma[a'l_1 + a^\sigma T(l')] + T(l'')[a'l_0^\sigma + l_1l_0' + a^\sigma a'']$
++++	$aa'a''l_0' + aa''^\sigma l_0^\sigma l_0' + aa''l_1l_0''^\sigma + al_0^\sigma T(l')l_0''^\sigma + l_0^\sigma T(l)l_0'^\sigma l_0''^\sigma + a^\sigma a''^2l_0 + a^\sigma a''l_1' + a^\sigma T(l)l_1''^\sigma + al_0'N(l') + a^{1+\sigma}l_0'l_1''^\sigma + a'a''l_1 + a'^\sigma l_0l_0'^2 + a'l_0^\sigma l_1' + a'^\sigma T(l)l_0''^\sigma + a''^\sigma l_0^\sigma T(l) + a''T(l)^\sigma l_0' + l_1l_0'l_1' + a'^{1+\sigma}l_0' + T(l)T(l')^\sigma + a'''^\sigma[aa'l_1 + a^\sigma a'^\sigma l_0 + a^\sigma T(l)l_0'^\sigma + l_1N(l) + a^{1+\sigma}T(l')] + T(l'')[N(l)^\sigma + a^\sigma a'^\sigma + aa'l_0^\sigma + al_1l_0' + a^{1+\sigma}a'']$
+-++	$al_0^\sigma N(l'')^\sigma + a''l_0^\sigma l_0'^\sigma l_0''^\sigma + aa''^2l_0''^\sigma + al_0'^2l_1''^\sigma + a'^\sigma a''l_0''^\sigma + a'l_0^\sigma l_1''^\sigma + a''N(l') + l_1l_0'l_1''^\sigma + a'a''^2 + a''^{1+\sigma}l_0^\sigma + a'''^\sigma[a'l_0^\sigma l_0'^\sigma + al_1'^\sigma + a'^{1+\sigma} + aa''^\sigma l_0^\sigma + l_1T(l')] + T(l'')[a'^\sigma l_0' + l_0^\sigma l_1' + a''l_1]$
++-+	$a^\sigma a''l_0'^\sigma l_0''^\sigma + a^\sigma l_0l_0'l_1''^\sigma + a'a''l_0l_0' + a''N(l)l_0''^\sigma + a^\sigma a'l_1''^\sigma + aa''^\sigma l_0'^2 + al_1'^\sigma l_0''^\sigma + a^{1+\sigma}N(l'')^\sigma + a'N(l') + a''^\sigma T(l)l_0' + l_0l_0'N(l') + T(l)T(l')l_0''^\sigma + a^\sigma a''^{1+\sigma} + a'^2a'' + a'''^\sigma[a^\sigma a'l_0'^\sigma + a'N(l) + aa'^2 + a^{1+\sigma}a''^\sigma + a^\sigma l_0T(l')] + T(l'')[a'T(l) + N(l)l_0' + a^\sigma l_1' + a^\sigma a''l_0]$
+--+	$a'l_0'l_1''^\sigma + a''^2l_0l_0''^\sigma + a''T(l')l_0''^\sigma + l_0l_0'^2l_1''^\sigma + T(l)N(l'')^\sigma + a''^{1+\sigma}l_0' + a'''^\sigma[l_0l_1'^\sigma + a''^\sigma T(l) + a'T(l')] + T(l'')[N(l') + a'a'']$

$\texttt{0001}$	$aa'l'_0l''^{\sigma}_1 + aa''^2l_0l''^{\sigma}_0 + aa''T(l')l''^{\sigma}_0 + al_0l'^2_0l''^{\sigma}_1 + aT(l)N(l'')^{\sigma}$ $+ a'^{\sigma}a''l_0l''^{\sigma}_0 + a''l_0N(l') + a''T(l)l'^{\sigma}_0l''^{\sigma}_0 + N(l)l'_0l''^{\sigma}_1 + aa''^{1+\sigma}l'_0$ $+ a'a''^2l_0 + a'^{\sigma}a''^{\sigma}l'_0 + a'a''l'_1 + a'T(l)l''^{\sigma}_1 + a'^{\sigma}T(l')l''^{\sigma}_0 + a''^{1+\sigma}T(l)$ $+ l'_1N(l') + a'''^{\sigma}[al_0l'^{\sigma}_1 + a'T(l)l'^{\sigma}_0 + a'^{1+\sigma}l_0 + aa''^{\sigma}T(l) + aa'T(l')$ $+N(l)T(l')]+T(l'')[a'^{1+\sigma}+a'^{\sigma}l_0l'_0+T(l)l'_1+aN(l')+a''N(l)+aa'a'']$

Appendix B: Erratum to [1]

This section serves as an erratum to formula (85) of [1, Lemma 6.3]. The value for $Q[4]$ should read

$$Q[4] \quad = \quad Q[2] + \sum_{\substack{u\in\Phi_S \\ \langle r_2,u\rangle=-1}} e[u]e[-u] \quad = \quad e[3]^2 + \sum_{\substack{u\in\Phi_S \\ \langle r_2,u\rangle=-1}} e[u]e[-u]$$

(the term $Q[2]$, or equivalently $e[3]^2$, is missing in the original).

Alternatively

$$Q[4] \quad = \sum_{\substack{u\in\Phi_S \\ \langle \texttt{1001},u\rangle=-1}} e[u]e[-u].$$

Part 3 of the proof of Lemma 6.3 states that $\langle r_3, r_j\rangle = 0$ except when $j = 4$, which is correct, but irrelevant. Instead we should look at $\langle r_j, r_3\rangle$ which is 1 when both $j = 2$ or 4, and hence $Q_{(r_3)}$ is equal to $Q[2] + Q[4]$ instead of $Q[4]$.

Note that $\langle r_j, \texttt{0001}\rangle = 0$ except when $j = 4$, and hence we may instead restrict the sum to those u for which $\langle \overline{\texttt{0001}}, u\rangle = \langle \texttt{1001}, u\rangle = -1$.

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